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# Some properties of Julia sets of transcendental entire functions with multiply-connected wandering domains

Masashi KISAKA (木坂 正史)

Department of Mathematical Sciences,

Graduate School of Human and Environmental Studies,

Kyoto University, Yoshida Nihonmatsu-Cho, Kyoto 606-8501, Japan

(京都大学大学院 人間・環境学研究科 数理科学講座)

e-mail : kisaka@math.h.kyoto-u.ac.jp

**Abstract**

We study Julia components of transcendental entire functions with multiply-connected wandering domains. Under the assumption that the post singular set is contained in the Fatou set, it is shown that every repelling periodic point $p$ satisfies either

(1) $C(p) \supset \partial U$, where $C(p)$ is the Julia component containing $p$ and $U$ is an immediate attractive basin.

(2) $C(p) = \{p\}$ and this is a buried singleton component of $J(f)$.

## §1 Introduction

Let $f$ be a transcendental entire function, $F(f)$ its Fatou set and $J(f)$ its Julia set. The following are some fundamental results on the connectivity of $J(f)$:

**Proposition 1** If every Fatou component is bounded and simply connected, then $J(f) \subset \mathbb{C}$ is connected.

So it follows that if $J(f) \subset \mathbb{C}$ is disconnected, then either

(a) $f$ has an unbounded Fatou component or

(b) $f$ has a multiply-connected Fatou component.

For the case (a), the following holds. Note that an unbounded Fatou component $U$ is always simply connected (see [Ba1]) and so we can consider a Riemann map $\varphi : \mathbb{D} \to U$ of $U$.

**Theorem 2 ([K, p.192, Main Theorem])** Suppose there exists an unbounded invariant Fatou component $U$ and let us consider the following conditions:

(A) $\infty \in \partial U$ is accessible in $U$.

(B) There exist a finite point $q \in \partial U$ with $q \notin P(f)$, $m_0 \in \mathbb{N}$ and a continuous curve $C(t) \subset U$ $(0 \leq t < 1)$ with $C(1) = q$ which satisfies $f^{m_0}(C) \supset C$, where

$$P(f) = \overline{\bigcup_{n=0}^{\infty} f^n(\mathrm{sing}(f^{-1}))}$$

is the post-singular set of $f$.

(1) If $U$ is either an attractive basin with (A) and (B), or a parabolic basin with (A) and (B), or a Siegel disk with (A), then the set

$$\Theta_\infty := \{e^{i\theta} \mid \varphi(e^{i\theta}) := \lim_{r \nearrow 1} \varphi(re^{i\theta}) = \infty\} \subset \partial\mathbb{D}$$

is dense in $\partial\mathbb{D}$. In particular, $J(f) \subset \mathbb{C}$ is disconnected.

(2) If $U$ is a Baker domain with (B) and $f|U$ is not univalent, then $\Theta_\infty$ is dense in $\partial\mathbb{D}$ or at least its closure $\overline{\Theta_\infty}$ contains a certain perfect set in $\partial\mathbb{D}$. In particular, $J(f) \subset \mathbb{C}$ is disconnected.

Next result is a generalization of the above result:

**Theorem 3 [BD1, p.439, Theorem 1.1, 1.2, Corollary 1.3]** Theorem 2 holds without the assumption (B).

On the other hand, $J(f) \subset \mathbb{C}$ can be connected nevertheless $f$ has an unbounded Fatou component. For example,

$$f(z) = 2 - \log 2 + 2z - e^z$$

has a Baker domain but $J(f)$ is connected ([**K**, p.194, Theorem 4]).

For the case (b), it is known that if $f$ has a multiply-connected Fatou component $U$, then $U$ is a wandering domain and bounded (see, [**Ba2**, Theorem 3.1]) and therefore $J(f) \subset \mathbb{C}$ is always disconnected. Furthermore $J(f) \cup \{\infty\} \subset \widehat{\mathbb{C}}$ is also disconnected and actually this is the only case where $J(f) \cup \{\infty\} \subset \widehat{\mathbb{C}}$ can be disconnected as follows:

**Proposition 4 ([K, p.191, Theorem 1])** $J(f) \cup \{\infty\} \subset \widehat{\mathbb{C}}$ is disconnected if and only if $f$ has a multiply-connected wandering domain.

In what follows, we will concentrate on the case (b), that is, the case where $f$ has a multiply-connected wandering domain $U$ and investigate some properties of connected components of the Julia set, which we call *Julia components.* We note the following fact (see, [**Ba2**, p.565, Theorem 3.1]):

**Proposition 5** If $U$ is a multiply-connected wandering domain, then $f^n|U \to \infty$.

**Definition 6** (1) We call a connected component of $J(f)$ a *Julia component.*

(2) $z \in J(f)$ is called a *buried point* if $z$ satisfies $z \notin \partial U$ for any Fatou component $U$.

(3) We call the set

$$J_0(f) := \{z \in J(f) \mid z \text{ is a buried point}\}$$

the *residual Julia set* of $f$.

(4) A Julia component $C$ is called a *buried component* if $C \subset J_0(f)$.

For rational cases, the following are known:

**Example 7** ([Mc]) Let $f(z) = z^2 + \dfrac{\lambda}{z^3}$, where $\lambda > 0$ is small. Then $J(f)$ is a Cantor set of nested quasi-circles. So there are buried components. In particular, $J_0(f) \neq \emptyset$.

**Theorem 8** ([Mo, p.208, Theorem 3]) Let $f$ be a hyperbolic rational function. Then $J_0(f) \neq \emptyset$ if and only if

(1) $F(f)$ has a completely invariant component, or

(2) $F(f)$ consists of only two components.

**Example 9** ([Mo, p.209]) Let $f(z) = \dfrac{-2z+1}{(z-1)^2}$, then the following hold:

(1) The set $\{0,\ 1,\ \infty\}$ is a super-attracting cycle.

(2) $f$ is hyperbolic.

(3) Any Fatou component is a preimage of the super-attractive basin above.

(4) $J(f)$ is connected.

So by Theorem 8, we have $J_0(f) \neq \emptyset$. But since $J(f)$ is connected, there is no buried component.

**Example 10** ([U]) There exists a rational function $f$ whose Julia set is homeomorphic to a Sierpinski gasket. So $J_0(f) \neq \emptyset$, but again there is no buried component.

Here are some fundamental properties for buried points and residual Julia sets. Note that $f$ need not be rational and these hold also for transcendental entire functions and even for meromorphic functions.

**Proposition 11** (1) If $F(f)$ has a completely invariant component, then $J_0(f) = \emptyset$.

(2) If there exists a buried component of $J(f)$, then $J(f)$ is disconnected.

(3) If $J_0(f) \neq \emptyset$, then $J_0(f)$ is completely invariant, dense in $J(f)$, and uncountable.

More information on residual Julia sets, see [DF].

## §2 Results

Main result of this paper is as follows:

**Theorem A** Let $f$ be a transcendental entire function. Assume that

(a) $P(f) = \overline{\bigcup_{n=0}^{\infty} f^n(\mathrm{sing}(f^{-1}))} \subset F(f)$,

(b) $f$ has a multiply-connected wandering domain.

Then every repelling periodic point $p$ satisfies either one of the following:

(1) $C(p) \supset \partial U$, where $C(p)$ is the Julia component containing $p$ and $U$ is an immediate attractive basin.

(2) $\{p\}$ is a buried singleton component of $J(f)$.

**Corollary B** Let $f$ be a transcendental entire function. Assume the above conditions (a), (b) and also

(c) $f^n(z) \to \infty$ for any $z \in F(f)$.

Then every repelling periodic point $p$ is a buried singleton component of $J(f)$.

**Remark** $f$ is called *hyperbolic* if

$$\mathrm{dist}_{\mathbb{C}}(P(f), J(f)) > 0,$$

where $\mathrm{dist}_{\mathbb{C}}$ is the Euclidean distance on $\mathbb{C}$. So the condition (a) in Theorem A is slightly weaker than hyperbolicity.

**(Outline of the Proof):** Let $p$ be a repelling periodic point. For simplicity, we assume that $p$ is a fixed point. Suppose that $p$ does not satisfy (1). Let $C(p) \subset J(f)$ be the Julia component containing $p$. Then $f(C(p)) = C(p)$ and we can show that $C(p)$ is bounded. If there exists a Fatou component $U \subset F(f)$ such that $C(p) \cap \partial U \neq \emptyset$, then it follows that $U$ is a wandering domain which satisfies $f^n(U) \to \infty$ $(n \to \infty)$. Then this contradicts the fact that $C(p)$ is bounded. Hence $C(p)$ is a buried component.

Next we can show that the complement of $C(p)$ has no bounded component. Then since $P(f) \subset F(f)$ and $C(p)$ is bounded, we have

$$\mathrm{dist}_{\mathbb{C}}(C(p), P(f)) > 0.$$

Then there exists a simply connected domain $W$ such that $C(p) \subset W$ and there exists a branch $g_n$ of $f^{-n}$ which satisfies $g_n(p) = p$. It is well-known that $\{g_n\}_{n=1}^{\infty}$ is a normal family and hence there exists a subsequence $g_{n_i}$ converging to a constant function which must be the point $p$. On the other hand, we have $g_n(C(p)) = C(p)$, so we conclude that $C(p) = \{p\}$. This completes the proof of Theorem A. Corollary B is an immediate consequence of Theorem A. □

# §3 Examples

**Example 12 ([BD2, p.375, Theorem G])** There exists an $f(z)$ with the following form

$$f(z) = k\prod_{n=1}^{\infty}\left(1+\frac{z}{r_n}\right), \quad 0 < r_1 < r_2 < \cdots, \; k > 0$$

such that for every repelling periodic point $p$ is a buried singleton component of $J(f)$.

**Example 13 ([KS])** There exists a transcendental entire function $f$ with doubly-connected wandering domains, which satisfies the following: Every critical point $c$ satisfies $f^2(c) = 0$ and 0 is a super-attracting fixed point. This implies that this $f$ satisfies the assumptions of Theorem A. Therefore every repelling periodic point $p$ satisfies either $C(p) \supset \partial U$ for the immediate attractive basin $U$ of the super-attractive fixed point 0 or $\{p\}$ is a buried singleton component of $J(f)$.

**Example 14 ([Be])** By using the similar method as in Example 13, Bergweiler constructed an example of transcendental entire function $f$ which has both a simply connected and a multiply connected wandering domain. Critical points of $f$ satisfy the following:

(1) $c_0 = 0 < c_1 < c_2 < \cdots \to \infty$,

(2) $f(0) = 0, \; f(c_i) = c_{i+1}, \; i = 1, 2, \ldots$

(3) $c_i$ is contained in a simply connected wandering domain $U_i$ which satisfies

$$f(U_i) = U_{i+1}, \; f^n|U_i \to \infty \; (n \to \infty).$$

So this $f$ also satisfies the assumptions (a) and (b) of Theorem A.

**Example C** We can construct an $f$ which satisfies the assumptions (a), (b) and (c) by using the similar method as in Example 13. Hence every repelling periodic point $p$ is a buried singleton component of $J(f)$ from Corollary B. We omit the details.

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